LA-UR -77-2671

TITLE: SPECTRAL ESTIMATION ACCURACY WITH ARBITRARY TRUNCATING FUNCTIONS

CONF-771206--19

MASTER

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SUBMITTED TO: Digital Signal Processing Symposium, Albuquerque, NM December 6-7, 1977

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Form No. 836
St. No. 2629
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UNITED STATES
ENERGY RESEARCH AND
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CONTRACT W-7405-ENG. 36

SPECTRAL ESTIMATION ACCURARY
WITH ARBITRARY TRUNCATING FUNCTIONS*

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ABSTRACT

A Fourier domain derivation is given for the bias and statistical stability of power spectral estimates of limited amounts of data. The data is represented as the product of a stationary stochastic process and a truncating function that is zero outside a specified region. The error estimates are derived for arbitrary truncating functions, and they agree with already known expressions when the truncating function is a rectangle. Comparisons of spectral estimate accuracies for rectangle and raised cosine truncations are given.

Many kinds of stationary processes can be characterized and studied by analyses of their Wiener spectra. Knowledge of spectral functions is applied to image processing problems in the Helstrom filtering,[1] in holomorphic filtering[2] and other restoration techniques, and it is often helpful in evaluating the effects of optical systems on specific kinds of images. Spectral calculations can be misleading, however, if attention is not given to the statistical errors that are associated with their calculation from specific data. The significance that is assigned to a Wiener spectral computation depends on the method used to control these errors.

Wiener spectra are ensemble averages, and the properties of individual members of the ensemble generally differ from their average. Meaningful analysis is carried out by computing properties whose standard deviations are small and which, nevertheless, are appropriate to the problem. For example, we rarely use periodogram estimates of spectral data but, instead, associate with a given frequency of a spectral function the average of several neighboring periodogram components, because the standard deviation of this quantity is smaller. The computation of this average is the well-known process of windowing.

*Work performed under the auspices of the U.S. Department of Energy, Contract No. W7405-ENG-36.

The selection of a section of data for analysis creates another kind of error. Power spectra are defined only for stationary processes, and a truncated process, which is zero outside some specified region and nonzero inside it, is not stationary. Finite segments of data are intended to approximate the infinite integration region of the Fourier integral, and the truncation error, often called bias, can be made smaller by choosing a larger segment of data. It may not always be possible to analyze an arbitrarily large segment, however. The image process may be locally stationary and treated as stationary only within limited regions, or there may simply not be very much data available. In cases such as these, it is important to estimate bias and to design measurement and computation procedures to minimize it.

Methods of estimating bias and random errors of spectral estimates are described in several reference works.[3,4] Derivations are rather long and are necessarily restricted to data that is limited by multiplication with a rectangular truncating function. This type of truncation function is not always appropriate for the processing of locally stationary images. It is often advantageous to segment such an image into stationary regions that overlap so that severe divisions between regions are not visible. When this is done, a raised cosine truncating function has proved to be more useful than a rectangle.

We will derive the bias and random errors of stationary data by considering Fourier domain quantities. As with direct domain derivations, we will assume only that the data is stationary and, to estimate the random error, that it is normally distributed. In the Fourier domain, the computation of the error estimates is relatively straightforward, and it is easily extended to new situations such as nonrectangular truncation functions.

STATISTICS OF TRUNCATED DATA

We represent a finite segment of a stationary process $i_A(x)$ as the product of the stationary process $i(x)$ and a truncating function $t(x)$ that is zero outside the finite region.

$$i_A(x) = i(x)t(x), \tag{1}$$

If the Fourier transforms of $i_A(x)$, $i(x)$ and $t(x)$ are $g_A(f)$, $g(f)$ and $\tau(f)$, the transform $g_A(f)$ of Eq. (1) is

$$g_A(f) = \int g(f_1)\tau(f - f_1)df_1 \quad . \tag{2}$$

The integral of the square of the magnitude of $g_A(f)$ over some small region of the frequency domain is the Wiener spectrum that might be measured at the center of that region. Usually, the integral is multiplied by a weighting function $Q(f)$, known as a window, so that the integral is replaced by a convolution. The measured Wiener spectrum is

$$p(f_0) = \int Q(f - f_0)|g_A(f)|^2 df \tag{3}$$

or, when we substitute Eq. (2) into Eq. (3)

$$p(f_0) = \int Q(f - f_0)\tau^*(f - f_1)\tau(f - f_1')g^*(f_1)g(f_1')df_1 df_1' df \quad . \tag{4}$$

In order to compute the bias of our estimate of $P(f)$, we compute the ensemble average of $P(f)$ of $p(f)$ in terms of $G(f)$, the true Wiener spectrum of $i(x)$. We recall that spectral components of a stationary process are delta-correlated, and their ensemble average can be expressed

$$\langle g^*(f_1)g(f_1')\rangle = G(f)\delta(f - f') \quad . \tag{5}$$

When we make use of Eq. (5) to compute the ensemble average of Eq. (4), the measured Wiener spectrum becomes

$$P(f_0) = \int Q(f - f_o)|\tau(f - f_1)|^2 G(f_1)df_1 df \quad . \tag{6}$$

We can estimate the degree of bias if we know $G(f_1)$ approximately.

The standard deviation $\sigma(f_0)$ of $P(f_0)$ is a measure of its statistical stability, the extent to which the Wiener spectrum of a segment of data differs from the average of an ensemble of similar segments. This quantity can be computed similarly to the Wiener spectrum average. We find the value of

$$\sigma^2(f_0) = \langle p^2(f_0)\rangle - \langle p(f_0)\rangle^2 \quad . \tag{7}$$

With the aid of Eq. (4), this can be expressed in terms of $g(f)$

$$\sigma^2(f) = \int Q(f' - f_0)Q(f - f_0)\tau(f' - f_2)\tau^*(f' - f_2')\tau^*(f - f_1)\tau(f - f_1')$$

$$\times \langle g(f_2)g^*(f_2')g^*(f_1)g(f_1')\rangle df_1 df_1' df_2 df_2' df\, df' - \langle p(f)\rangle^2 \ . \quad (8)$$

If we assume that the process i(x) is Gaussian, the real and imaginary parts of its Fourier transform are normally distributed. A theorem that relates ensemble averages of products of normally distributed quantities[5] leads to the relationship

$$\langle g(f_2)g^*(f_2')g^*(f_1)g(f_1')\rangle = \langle g(f_2)g^*(f_2')\rangle\langle g^*(f_1)g(f_1')\rangle$$
$$+ \langle g(f_2)g^*(f_1)\rangle\langle g^*(f_2')g(f_1')\rangle \quad (9)$$
$$+ \langle g(f_2)g^*(-f_1')\rangle\langle g^*(f_2')g(-f_1)\rangle \ .$$

To compute the last term of Eq. (9), we have made use of the relationship between positive and negative frequency portions of the Fourier transform of a real function

$$g(f) = g^*(-f) \ . \quad (10)$$

When we substitute Eq. (9) into Eq. (8) and use Eq. (5) to express the ensemble averages in terms of the Wiener spectrum G(f), the expression for the variance $\sigma^2(f)$ becomes

$$\sigma^2(f) = \int Q(f' - f_0)[Q(f - f_0) + Q(f + f_0)]\tau(f' - f_1)\tau^*(f' - f_1') \quad (11)$$
$$\times\ \tau^*(f - f_1)\tau(f - f_1')G(f_1)G(f_1')df_1 df_1' df\, df' \ .$$

Often the truncating function is a rectangle and the Fourier transform $\tau(f)$ is a sinc function. When this substitution is made in Eq. (11), the expression is identical to the known expression for a spectral estimate that is derived in the direct domain.[6]

We should observe from Eq. (6) that the weighting function Q(f) and the square of the Fourier transform of the truncating function affect the bias in the same way, but that their effects on $\sigma(f)$ are different. The two functions are often referred to interchangeably as "windows." They should not be; the overall role of the spectral window $Q(f)$ is different from that of the function $|\tau(f)|^2$.

The effects of windowing and truncation may be easily understood from a heuristic point of view when viewed in the Fourier transform domain. We see from Eq. (5) that spectral components of a stationary process are statistically independent and from Eq. (9) that components of the Wiener spectrum at different frequencies are likewise independent. Moreover, the variance of a spectral extimate is comparable to its mean. The variance of a windowed spectral estimate is less because this estimate is a sum of independent quantities. When a process is truncated, however, its spectral components are not independent, but correlated. This can be shown easily from Eq. (2); the correlation coefficient between $g_A(f)$ at different frequencies is, using Eq. (5)

$$\langle g_A^*(f) g_A(f')\rangle = \int G(f_1)\tau^*(f - f_1)\tau(f' - f_1)df_1 \quad . \tag{12}$$

Sums of truncated spectral components, which are functions of $g_A(f)$, may not have a smaller variance because of the lack of statistical independence of the summands. They are more highly correlated when the truncated segment is smaller; i.e., when the sinc function described by $\tau(f)$ is broader if the truncation function is a rectangle.

The equivalent degrees of freedom is often used as a measure of the statistical stability of a spectral measurement. This is the ratio $2P^2(f_0)/\sigma^2(f_0)$, which is sometimes the number of statistically independent spectral components that determines the estimate $P(f_0)$. Analytical expressions for the equivalent degrees of freedom can be derived for the asymptotic limits of small and large L, if we assume a rectangular truncation function. Then its Fourier transform $\tau(f)$ is givey by

$$\tau(f) = L \text{ sinc } \pi f L. \tag{13}$$

We will also assume that the window weighting function $Q(f)$ is a rectangle of width Δf centered at f_0. Equation (11) may be written

$$\sigma^2(f_0) \simeq 2L^4 \int_{\Delta f} df\,df' \left\{ \left[\int_{-\infty}^{\infty} \mathrm{sinc}[\pi(f' - f_1)L]\mathrm{sinc}[\pi(f - f_1)L]\; G\,(f_1)df_1 \right]^2 \right.$$

$$\left. + \left[\int_{-\infty}^{\infty} \mathrm{sinc}\;[\pi(f' - f_1)L]\;\mathrm{sinc}\;[\pi(f + f_1)L]G(f_1)df_1 \right]^2 \right\} \quad . \qquad (14)$$

If L is large, the central lobe of the sinc function is narrow, and the integral of the inner integrals is significant only when f',f_1 and f or $-f'$ are approximately equal. We will assume that $G(f_1)$ varies little over a region in frequency space of 4/L, twice the width of the central lobe, and we will remove this function from the inner integral. The expression for $\sigma^2(f_0)$ becomes

$$\sigma^2(f_0) \simeq 2L^2 \int_{\Delta f} df\;df' G^2(f)\mathrm{sinc}\;[\pi(f - f')L] \quad . \qquad (15)$$

If the central lobe of the sinc function is much smaller than Δf, we can replace the integration limits of f' by $(-\infty,\infty)$ and simplify further

$$\sigma^2(f_0) \simeq 2L \int G^2(f)df_1 \qquad (16)$$

when $L\Delta f >> 1$ and G(f) vary little over a region of size 4/L.

An expression for the ensemble mean spectral estimate can also be derived for the case when Q(f) is a rectangle and $\tau(f)$ is a sinc function Eq. (6) becomes

$$P(f_0) = L \int_{\Delta f} G(f)df \;. \qquad (17)$$

The equivalent degrees of freedom can be determined from Eqs. (15 and 16) to be

$$\mathrm{E.D.F.} = L\left[\int_{\Delta} G(f)df \right]^2 \Big/ \int_{\Delta f} G^2(f)df \quad , \qquad (18)$$

and if, as is often assumed, the best a priori estimate of the Wiener spectrum of G(f) is a constant, Eq. (17) becomes

$$\text{E.D.F.} = L\Delta f \ , \tag{19}$$

a well-known result.

The Wiener spectrum of many pictorial images is not a constant, but can be several orders of magnitude greater at low frequencies than at high ones. The equivalent degrees of freedom is smaller than for constant G(f), because spectral measurements are dominated by a small part of the spectrum. This lack of statistical stability is perhaps behind the frequently heard assertion that images are "not stationary." There is often no reason to assume, on an a priori basis, that the statistics of an image differ from place to place, and without such a priori knowledge a maximum ignorance assumption of stationarity should be made. It should be kept in mind, however, that, for commonly encountered image sizes L and Wiener spectra G(f), many statistical measurements may be unstable.

An expression similar to Eq. (17) can be derived for very small L, i.e., for $L\Delta f \ll 1$. In this limit, the central lobe of the sinc functions is broad and integrations are performed under the assumption that they vary little over regions Δf in size. This leads to the expected result that there is one degree of freedom in the limit $L \rightarrow 0$. Fourier components are highly correlated when L is small, and the statistics of a sum differ little from that of a single component.

EXAMPLES

We have numerically compared bias and equivalent degrees of freedom for rectangular and raised cosine truncating functions t(x). The raised cosine is defined

$$\begin{aligned} t(x) &= \frac{1}{2}\left(1 + \cos \frac{\pi x}{L}\right) && |x| < L \\ &= 0 && |x| > L \quad . \end{aligned} \tag{20}$$

The parameter L that characterizes the size of the truncated region is the distance between points for which t(x) is one-half. Data must be included, however, over distances of 2L, although the part of the data function i(x) that is multiplied by the "tails" of t(x) is greatly deemphasized. The raised cosine truncating function provides a means for nonzero i(x) to drop gradually to zero, avoiding the abrupt change that occurs when it is multiplied by a rectangle function, and the spectral estimation errors created by this abrupt change, especially for higher frequencies, should be smaller.

Figure 1 is a measure of the measurement error, defined by

$$\text{M.E.} = \frac{P(f_0) - \int_{\Delta f} G(f)df}{\int_{\Delta f} G(f)df} \tag{21}$$

for the two kinds of truncating functions. In all computations, the spectral window Q(f) is a rectangle with a width Δf of 2, and the size of the truncation region L varies from 0 to 1. This means the size-bandwidth product $L\Delta f$, the abscissa of the graph, varies from 0 to 2. The true value of the Wiener spectrum G(f) is an exponential, given by

$$G(f) = \exp\left(-\frac{f^2}{50}\right) , \tag{22}$$

and the measurement error is calculated at frequency f_0 of 10.

The error curve associated with the raised cosine truncation function is marked by asterisks. It is considerably smaller than the error curve associated with the rectangular truncation function when the truncation parameter L is small. In Fig. 2 the equivalent degrees of freedom, computed from Eqs. 6, 11, and 16, are plotted against the size-bandwidth parameter $L\Delta f$. The stability associated with the cosine truncation function is characteristic of a data extent of 2L, even though most of the data is taken from regions half that long.

In many computations, the spectral window Q(f) is chosen so that the bias or statistical stability of the computation is within some desired bound. We should keep in mind, however, that the truncation function t(x) may also be used as a design parameter. We have seen that an alternative truncation function can provide smaller bias and greater statistical stability than the traditional rectangular function when it is possible to gradually, rather than abruptly, deemphasize the data outside a specified region. This may be useful in computing spectral estimates of locally stationary data.

PROFESSIONAL BIOGRAPHY

Martin Lahart received the BSEE from Princeton University in 1960, the MS in Physics from the University of Michigan in 1962, and the PhD in Optical Sciences from the University of Arizona in 1975. Between 1962 and 1971 he worked primarily on industrial applications of optics. He has been a staff member of the Los Alamos Scientific Laboratory since 1975. He is a member of the Optical Society of America.

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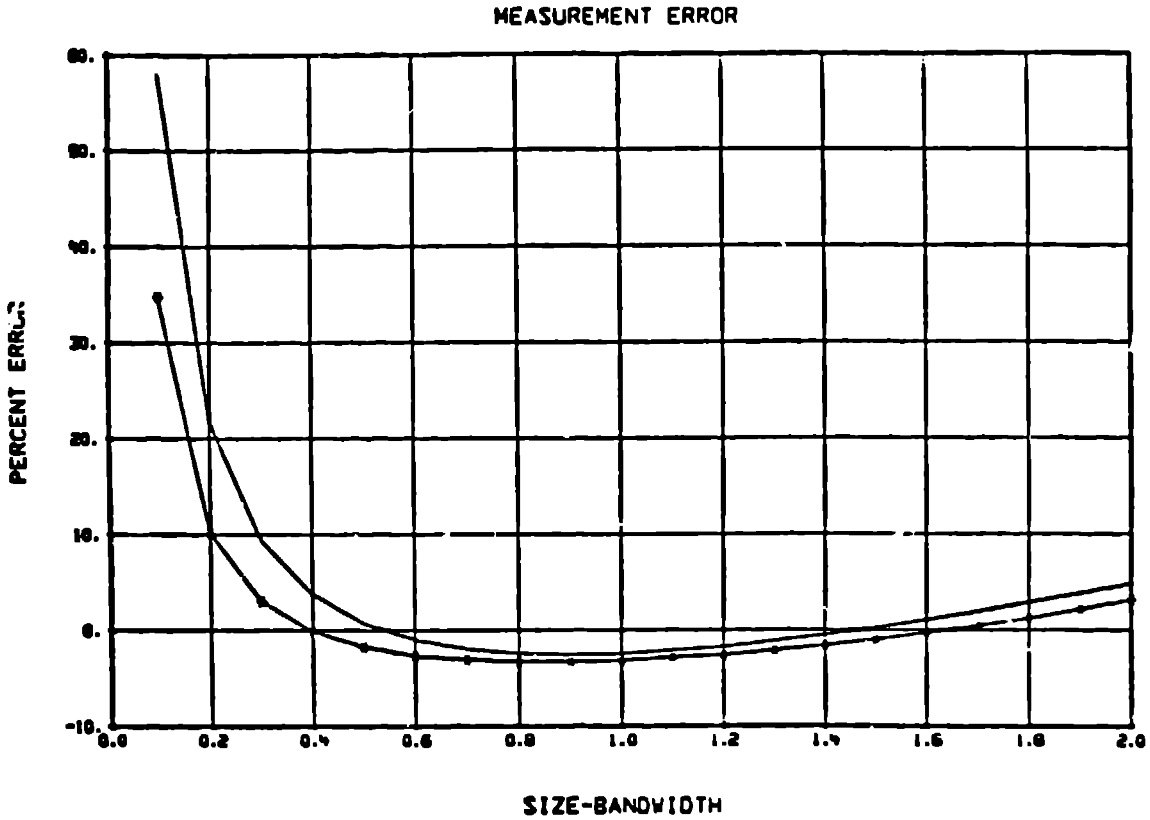

Fig. 1. Comparison of bias errors for rectangle (plain line) and raised cosine (asterisk line) truncating functions. The bandwidth Δf is 2 and the size varies from 0 to 1.

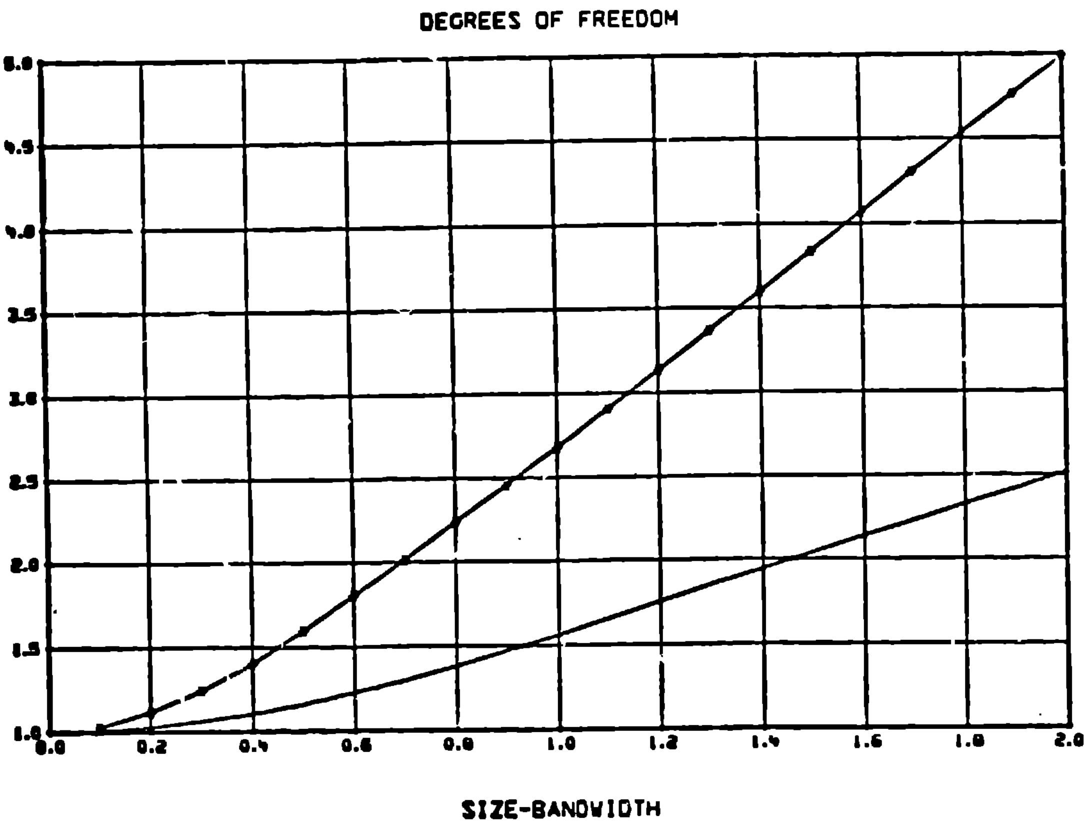

Fig. 2. Comparison of statistical stability of a spectral measurement with rectangular (plain line) and raised cosine (asterisk line) truncating functions.